Applications of Integration

Areas between Curves

- If $f(x) \geqslant g(x)$ on $[a, b]$, then the area between $y = f(x)$ and $y = g(x)$ on $[a, b]$ is
$$A = \int_a^b [f(x) - g(x)]\, dx$$
- If $f(y) \geqslant g(y)$ on $[c, d]$, then the area between $x = f(y)$ and $x = g(y)$ on $[c, d]$ is
$$A = \int_c^d [f(y) - g(y)]\, dy$$

Volumes with Cross Sections

If $A(x)$ is a solid's cross-sectional area at x, then the solid's volume between $x = a$ and $x = b$ is
$$V = \int_a^b A(x)\, dx$$

Disk Method

Consider a region that is adjacent to an axis of revolution. Let function r be the distance between the axis of revolution and a point on the graph.

- The volume generated by rotating the region about a horizontal line, $a \leqslant x \leqslant b$, is
$$V = \pi \int_a^b [r(x)]^2\, dx$$
- The volume generated by rotating the region about a vertical line, $c \leqslant y \leqslant d$, is
$$V = \pi \int_c^d [r(y)]^2\, dy$$

Washer Method

Draw an approximating rectangle *perpendicular* to the axis of revolution. Let r_{out} be the distance between the axis of revolution and the rectangle's farthest side, and let r_{in} be the distance between the axis of revolution and the rectangle's closest side.

- The volume generated by rotating the region about a horizontal line, $a \leqslant x \leqslant b$, is
$$V = \pi \int_a^b \left([r_{\text{out}}(x)]^2 - [r_{\text{in}}(x)]^2\right) dx$$
- The volume generated by rotating the region about a vertical line, $c \leqslant y \leqslant d$, is
$$V = \pi \int_c^d \left([r_{\text{out}}(y)]^2 - [r_{\text{in}}(y)]^2\right) dy$$

Shell Method

Draw an approximating rectangle *parallel* to the axis of revolution. Let r be the distance between the rectangle and the axis of rotation, and let h be the height (or length) of the rectangle.

- The volume generated by rotating the region about a vertical line, $a \leqslant x \leqslant b$, is
$$V = 2\pi \int_a^b r(x)h(x)\, dx$$
- The volume generated by rotating the region about a horizontal line, $c \leqslant y \leqslant d$, is
$$V = 2\pi \int_c^d r(y)h(y)\, dy$$

Work

The work required to move an object from $x = a$ to $x = b$ is
$$W = \int_a^b F(x)\, dx$$
where $F(x)$ is the variable force.

Springs and Gases

- To stretch a spring of stiffness k a distance x, a force $F(x) = kx$ must be applied.
- A gas of pressure P and volume V at a constant temperature satisfies $PV = C$, where C is a constant, and the work done by the gas is
$$W = \int_{V_a}^{V_b} \frac{C}{V}\, dV$$
where V_a and V_b are the initial and final volumes, respectively.

Average Value of a Function

If f is continuous on $[a, b]$, then the average value of f on $[a, b]$ is
$$f_{\text{avg}} = \frac{1}{b - a} \int_a^b f(x)\, dx$$

Mean Value Theorem for Integrals

If f is continuous on $[a, b]$, then a number c exists in $[a, b]$ such that $f(c) = f_{\text{avg}}$:
$$f(c) = \frac{1}{b - a} \int_a^b f(x)\, dx$$

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Applications of Integration

Arc Length

If a function has a continuous derivative on $[a, b]$, then the arc length on $[a, b]$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Surface Area of Revolution

- Revolving about the x-axis (where $y > 0$), $a \leqslant x \leqslant b$:

$$S = \int_a^b 2\pi y\, ds$$

- Revolving about the y-axis (where $x > 0$), $c \leqslant y \leqslant d$:

$$S = \int_c^d 2\pi x\, ds$$

where

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{or} \qquad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Consumer Surplus

The area of the region bounded by the demand curve $p = D(x)$ and the market price $p = P$, where X is the quantity demanded, is

$$s_C = \int_0^X [D(x) - P]\, dx$$

Producer Surplus

The area of the region bounded by the supply curve $p = S(x)$ and the market price $p = P$, where X is the quantity supplied, is

$$s_P = \int_0^X [P - S(x)]\, dx$$

Moments and Centers of Masses (Particles)

Consider a series of particles of masses $m_1, m_2, \ldots, m_n$ located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

- Moment about the x-axis:

$$M_x = \sum_{i=1}^{n} m_i y_i$$

- Moment about the y-axis:

$$M_y = \sum_{i=1}^{n} m_i x_i$$

- Center of mass:

$$\overline{x} = \frac{M_y}{m} \qquad \overline{y} = \frac{M_x}{m}, \qquad m = \sum_{i=1}^{n} m_i$$

Moments and Centroids (Laminae)

Consider a region bounded by the graphs of $y = f(x)$ and $y = g(x)$, $f(x) \geqslant g(x)$, $a \leqslant x \leqslant b$, with uniform surface density ρ.

- Moment about the x-axis:

$$M_x = \rho \int_a^b \tfrac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx$$

- Moment about the y-axis:

$$M_y = \rho \int_a^b x\,[f(x) - g(x)]\, dx$$

- Centroid:

$$\overline{x} = \frac{1}{A}\int_a^b x\,[f(x) - g(x)]\, dx$$
$$\overline{y} = \frac{1}{A}\int_a^b \tfrac{1}{2}\left([f(x)]^2 - [g(x)]^2\right) dx$$

where A is the region's area.

Theorem of Pappus

Suppose that a region has an area A and centroid $(\overline{x}, \overline{y})$ and lies entirely on one side of an axis of revolution.

- Volume produced by rotating the region about the x-axis:

$$V = 2\pi A\,|\overline{y}|$$

- Volume produced by rotating the region about the y-axis:

$$V = 2\pi A\,|\overline{x}|$$

Hydrostatic Force

If a vertical plate is submerged in water between the levels $y = a$ and $y = b$ such that at some y, $h(y)$ is the depth beneath the surface and $w(y)$ is the body's width, then the hydrostatic force acting upon the body is

$$F = \gamma \int_a^b h(y) w(y)\, dy$$

In SI units, $\gamma = 9800\,\mathrm{N/m^3}$; in imperial units, $\gamma = 62.5\,\mathrm{lb/ft^3}$.

Probability Density Functions

If $f(x)$ is a probability density function, then

$$f(x) \geqslant 0 \qquad \int_{-\infty}^{\infty} f(x)\, dx = 1$$

The probability of a random variable X taking on a value between a and b is

$$P(a \leqslant X \leqslant b) = \int_a^b f(x)\, dx$$

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Applications of Integration

Mean of a Probability Density Function

If $f(x)$ is a probability density function, then the mean is

$$\mu = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x$$

Exponential Probability Density Functions

If $k > 0$, then an exponential probability density function takes the form

$$f(t) = \begin{cases} 0 & t < 0 \\ ke^{-kt} & t \geqslant 0 \end{cases}$$

where the mean is $\mu = \dfrac{1}{k}$.

Normal Distributions

The probability density function of a Normal distribution is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/\left(2\sigma^2\right)}$$

where μ is the mean and σ is the standard deviation.

By the 68–95–99.7 rule, 68% of data reside within 1 standard deviation of the mean, 95% reside within 2 standard deviations, and 99.7% reside within 3 standard deviations.

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